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Exam II: MTH 213, Spring 2018

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Score = $\frac{33}{33}$

QUESTION 1. (6 points)

Imagine: The following Algorithm segment. Find the exact number of additions, multiplications, and subtractions that will be performed when the algorithm is executed. Then find the complexity of the Algorithm segment.

$m = 7; s = 3$

For $k := 1$ *to* $\lfloor \frac{n}{2} \rfloor$

$L = k * m + s^2 + 2 * s - 2$

For $i := 1$ *to* $(k + 2)$

$W = s^4 + 3 * k + m^3 + i - 8$

next i

next k

* Assume n is odd.

Outer loop:

∘ Iterates from $k = 1$ to $\lfloor \frac{n}{2} \rfloor$

If n is odd: $\lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}$

~~If n is even: $\lfloor \frac{n}{2} \rfloor = \frac{n}{2}$~~

∘ Total number of times code is executed:

~~Even: $\frac{n}{2} - 1 + 1 = \frac{n}{2}$~~

Odd: $\frac{n-1}{2} - 1 + 1 = \frac{n-1}{2}$

∘ Number of operations per single iteration of outer loop = 6

$[L = k * m + s * s + 2 * s - 2]$

∘ Total number of operations in outer loop:

~~$= 6(\frac{n}{2}) = 3n$ [Even]~~.

odd $= 6(\frac{n-1}{2}) = \boxed{3n-3}$.

Inner loop:

∘ Iterates from $i = 1$ to $(k+2)$.

∘ Number of operations per single iteration of inner loop = 10

$[s * s * s * s + 3 * k + m * m * m + i - 8]$

∘ Number of ~~iterations~~ operations of inner loop for each outer loop:

$k=1$, $k=2$, $k=3$, $k = \frac{n-1}{2}$

$\underbrace{10 \times 3,\ 10 \times 4,\ 10 \times 5, \ldots, 10 \times (\frac{n-1}{2} + 2)}_{\text{arithmetic}}$

$= \frac{(\frac{n-1}{2})[(10 \times 3) + (10(\frac{n-1}{2} + 2))]}{2}$

$= \frac{(\frac{n-1}{2})(5n + 45)}{2}$

$= \frac{5n^2 + 40n - 45}{4}$.

not need

∴ Total operations for algorithm

$= \boxed{\frac{5n^2 + 40n - 45}{4} + 3n - 3}$

$[= S_n]$

∴ complexity $= O(S_n) = \underline{n^2}$.

QUESTION 2. (2 points)

(i) $O(\frac{x^{0.9} + 3x^{1.2} - 5}{x + 7})$ equals to

$\frac{O(x^{0.9} + 3x^{1.2} - 5)}{O(x+7)} = \frac{n^{1.2}}{n^1} = n^{1.2-1} = \underline{x^{0.2}}$

(ii) $O(\sqrt[5]{x} + x^2 + 1)(x^2 - x^{7/2} - 2)$ equals to

$O(\sqrt[5]{x} + x^2 + 1) \cdot O(x^2 - x^{7/2} - 2) = n^2 \cdot n^{7/2} = n^{2 + 7/2} = \underline{n^{11/2}}$.

QUESTION 4. (6 points)
Use math induction to convince me that $12 \mid (5^{4m} - 1)$, $m \geq 1$.

$\boxed{1}$ Prove it for $m = 1$:

when $m = 1$:

$5^{4(1)} - 1 = 5^4 - 1$
$= 625 - 1$
$= 624$

$\frac{624}{12} = 52.$

$\therefore 12 \mid 5^4 - 1.$

$\boxed{2}$ Assume claim is valid for some $n = m \geq 1$:

$\Rightarrow 12 \mid 5^{4n} - 1$

$\boxed{3}$ Prove it for $(n+1)$:

[ie. we must show that $12 \mid 5^{4(n+1)} - 1$].

$5^{4(n+1)} - 1 = 5^{4n+4} - 1$
$= 5^{4n} \cdot 5^4 - 1$
$= 5^{4n} \cdot 5^4 - 5^4 + 5^4 - 1$
$= \underbrace{5^4(5^{4n} - 1)}_{\text{Divisible by 12, as shown in } \boxed{2}} + \underbrace{5^4 - 1}_{\text{Divisible by 12, as shown in } \boxed{1}}$

$\therefore 12 \mid 5^4(5^{4n} - 1) + 5^4 - 1.$

ie. $12 \mid 5^{4(n+1)} - 1.$ Hence $12 \mid 5^{4m} - 1$. QED.

QUESTION 5. (5 points) Let $A = \{\{3\}, 3, \cancel{3}, 5, \{3,5\}, \{\phi\}, \{6\}, \{6, x\}, x, 7\}$ and $B = \{3, \{3,5\}, x, \{7\}, \{3\}\}$. Then Write T or F

(i) $\{3\} \in A \cap B$ $\longrightarrow$ T.

(ii) $7 \in A - B$ $\longrightarrow$ T.

(iii) $\{\phi\} \in A - B$ $\longrightarrow$ T.

(iv) $\{\phi\} \subset A - B$. $\longrightarrow$ F. [$\{\{\phi\}\} \subset A - B$ would be true].

(v) $\{3, \phi\} \subset A$ $\longrightarrow$ F. $\phi \not\subset A$

(vi) $\{3,5\} \in A$ $\longrightarrow$ T.

(vii) $\{3,5\} \subset A$ $\longrightarrow$ T.

(viii) $B - A = \phi$ $\longrightarrow$ F. $B - A = \{\{7\}\}$.

(ix) $|AXB| = 13$ $\longrightarrow$ F.

(x) $\{\{3\}, x\} \subset A$ $\longrightarrow$ T.

~~Ans~~ $|A \times B| = |A| \times |B| = 9 \times 5 = 45$

QUESTION 6. (8 points) Let $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$. Define $=$ on A, where if $a, b \in A$, then $a = b$ if $(a + (12 - b))\ (mod\ 12) \in \{0, 4, 8\}$

(i) Convince me that $=$ is an equivalence relation on A.

Since 12 (mod 12) = 0, obseve tht (a + 12 - b) (mod 12) = (a - b) (mod 12). Let B = {0, 4, 8}

Remark: (1) note that (c + d) (mod 12) is in B for every c, d in B.

(2) note that in general If L (mod n) = k and k not = 0, then -L (mod n) = n - k

(A-A): Let a in A. Then (a - a) (mod 12) = 0 in B. Thus a = a for every a in A

(A-B): Assume that a = b for some a, b in A. We show b = a. We may assume that a, b are different elements in A. Since a = b and b is not equal to a, (a - b) (mod 12) = 4 or 8. If (a - b) (mod 12) = 4, then (b - a)(mod 12) = 12 - 4 = 8 in B (see 2 above). Thus b = a. If (a - b) (mod 12) = 8, then (b - a) (mod 12) = 12 - 8 = 4 in B. Hence, again, b = a.

(A-B-C): Assume that a = b and b = c for some a, b, and c in A. Hence (a - b) (mod 12) is in B and (b - c) (mod 12) is in B. Let n = (a - b) (mod 12) and m = (b - c) mod (12). Note that n, m are in B. Then (a - c) (mod 12) = [(a - b) + (b - c)] (mod 12) = (n + m) (mod 12) is in B by (1). Thus a = c

$(a + 12 - b + b + 12 - c)(mod\ 12)$

$(1 + 12 - 5 + 5 + 12 - 9)(mod\ 12) = 16$

$16\ (mod\ 12) = 16$

$(16 - 16)(mod\ 12) = 0.$

$(0)(mod\ 12) = 0.$

$0 \in \{0, 4, 8\}$. $\therefore a \text{ "="} c.$

$\therefore$ Axiom 3 holds.

$\therefore$ "=" is an equivalence relation.

(ii) Find all equivalence classes of $(A, =)$.

$[0] = \{0, 4, 8\}.$ ✓

$[1] = \{1, 5, 9\}.$ ✓

$[2] = \{2, 6, 10\}.$ ✓

$[3] = \{3, 7, 11\}.$ ✓

(iii) view $=$ as a subset of $A \times A$. How many elements does $=$ have? Do not write down all elements of $=$

$$\text{"="} = ([0] \times [0]) + ([1] \times [1]) + ([2] \times [2]) + ([3] \times [3]).$$
$$= 3^2 + 3^2 + 3^2 + 3^2$$
$$= \underline{\underline{36}}.$$ ✓

Faculty information

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